Headsolving competition

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October 2023

Acknowledgements

Thank you to my friend Kevin for testing these out. Qualifiers

1. (Answer: 0.5) Integrate

$$\int_0^{\frac{\pi}{2}} \cos(x) \sin(x) \, \mathrm{d}x$$

- 2. (Answer: 504) How many ways can you arrange 1 blue book, 3 red books, and 5 yellow books
- 3. (Answer: $(x^2 + 1)(x + 2)$) Factorise $x^3 + 2x^2 + x + 2$ into (not necessarily linear) real components
- 4. (Answer: $\ln(3), \ln(5)$) Solve $e^{2x} 2e^x 15 = 0$
- 5. (Answer: $f^{-1}(x) = \frac{2-2e^x}{e^x+1}$) What is the inverse of $f(x) = \ln\left(\frac{2-x}{2+x}\right)$?
- 6. (Answer: $3 \times 53 \times 7$) What is the prime factorisation of 1113?
- 7. (Answer: $\frac{\sqrt{1-x^2}}{x}$) Simplify $\tan(\arccos(x))$
- 8. (Answer: 825) What is the product of the 5th fibonacci number (starting with "1,1..."), 5th prime and 5th triangle number?
- 9. (Answer: $17^2 + 5^2$) Express 314 as the sum of two squares
- 10. (Answer: 15°) What are the interior angles of a regular dodecagon?
- 11. (Answer 3.3) On a number line, if I start at 0 and move to 1 at 0.2 units per second, and you move from 1 to 0 at 0.1 units per second, how long will it take for us to pass by each other?
- 12. (Answer: 1652) Round 1653 to the nearest multiple of 7.
- 13. (Answer: (x-6)(x-5)(x+2)) Factorise $x^3 + 3x^2 28x 60$ given (x-6) is a factor.
- 14. (Answer: $x^4 9x^3 + x^2 + 81x + 70$) Expand (x+2)(x-5)(x+1)(x-7)
- 15. (Answer: 2.6) What is $\sqrt{7}$ to 1 decimal place?
- 16. (Answer: 32) What is the next term of the quadratic sequence starting with 5, 10, 19...
- 17. (Answer: 6) What is the last digit of 2^{284} ?
- 18. (Answer: y = -1) What is the horizontal asymptote of $f(x) = \frac{x-5}{7-x}$?
- 19. (Answer: $-\frac{1}{5040}$) What is the coefficient (as a number) of x^7 in the Maclaurin expansion for $\sin(x)$?

- 20. (Answer: 113) Convert 23 to base 4.
- 21. (Answer: 5) How many primes are between 70 and 90?
- 22. (Answer: $\frac{36}{35}$) Evaluate:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left(\frac{1}{4}\right)^n$$

- 23. (Answer: 23) What is the highest common factor between 161 and 299
- 24. (Answer: x = e) At what x does $f(x) = \frac{\ln(x)}{x}$ reach its maximum?
- 25. (Answer: $1 + \sqrt{2}$) Simplify $\sqrt{3 + 2\sqrt{2}}$
- 26. (Answer: 15) What is $\binom{6}{4}$?
- 27. (Answer: 36) What is the second number that is both a square and a triangle number?
- 28. (Answer: 12) How many 0s does 50! end in?
- 29. (Answer: 0.14) What are the first 2 decimal digits in the decimal expansion of $\frac{1}{7}$?
- 30. (Answer: $\frac{3}{111}$) What is 0.027027027... as a simplified fraction?
- 31. (Answer: $\frac{\pi}{2}$) What is $\arctan\left(\frac{1}{\sqrt{3}}\right) + \arctan\left(\sqrt{3}\right)$
- 32. (Answer: 1, 125, 8) What 3 positive cubes sum to 134?
- 33. (Answer: $\frac{1\pm\sqrt{1+4x}}{2}$) Simplify $\sqrt{x+\sqrt{x+\sqrt{x+\dots}}}$. Ignore issues of convergence.
- 34. (Answer: 16) How many equilateral triangles side length 1 does it take to tile the surface of a regular tetrahedron side length 2.
- 35. (Answer: 1260°) What is the sum of the interior angles of a nonagon?
- 36. (Answer: $\frac{1}{2\sqrt{2}}(1-\sqrt{3})$) What is $\cos\left(\frac{7\pi}{12}\right)$ in radicals?
- 37. (Answer: 190) Sum all natural numbers up to 19.
- 38. (Answer: 84) What is the area of a triangle with side lengths 7, 24, and 25?
- 39. (Answer: $\frac{1}{4}$) I have a unit square and connect the midpoints to form a new square. I connect the midpoints of that square to form another. What is the area of the final square?
- 40. (Answer: 24) What is the largest number of unit equilateral triangles that can tile a parallelogram with angles 60° and 120° with parallel sides of length 4 and a slope of 3.

Quarter-finals

- 1. (Answer: 65) Give a 2 digit number that can be written as the sum of 2 distinct positive squares in 2 ways.
- 2. (Answer: 10) How many ways can you tile a 4 by 4 square grid with only L shaped tetrominoes [4 connected squares (edge to edge) in an L shape] (you can flip them and the board is fixed in orientation).
- 3. (Answer: 15) By drawing lines between integer coordinates, on a 3×3 grid, how many ways can I draw a square?
- 4. (Answer: 8) How many integer solutions does $x^2 24 = y^2$ have?
- 5. (Answer: $y = \frac{x}{2} + \frac{c}{x}$) Solve $\frac{dy}{dx} = 1 \frac{y}{x}$?
- 6. (Answer: $y = e^{-1-e}x + 1 \frac{1}{e}$) What is the equation, in the form y = mx + c of tangent to the curve of $\ln(\ln(x))$ at $x = e^{e}$?
- 7. (Answer: 15th) What is the first power of 2 ending in 68?
- 8. (Answer: 10) A currency has coin values of 1, 3, 6, and 10. What is the smallest number of coins required to get 85p?

Semi-finals

- 1. (Answer: $2^n 2^{2^n}$) Given $x_0 = 2$, $x_{n+1} = 2x_n^2$, find x_n .
- 2. (Answer: 2) Find $\sum_{n=0}^{\infty} n2^{-n}$.
- 3. (Answer: $\frac{\sqrt{3}}{2} \frac{\sqrt{3}}{2\sqrt{2}}$) A unit equilateral triangle is sitting on its base. How high up should I make a horizontal cut to split it into two pieces of equal area?
- 4. (Answer: $\frac{1}{4}$) If I roll a fair 6-sided dice repeatedly. What is the chance of rolling 2 sixes before rolling any 1?.
- 5. (Answer: $Ae^{-x} + Be^{2x} + \frac{1}{12}e^x + \frac{x}{2}e^{-x}$) What is the general solution to the SODE $f''(x) + 3f'(x) + 2f(x) = \cosh(x)$)
- 6. (Answer: $xe^x + e^x$) Evaluate $\sum_{n=2}^{\infty} \frac{n^2 x^{n-1}}{n!}$

Finals

- 1. (Answer: $\sin(\sqrt{x})$) Find a divergent bounded sequence a_n (closed form (i.e no integrals or sums)) such that $a_n a_{n-1} \to 0$
- 2. (Answer: 3.5) In the domain [-1, 1], what is the maximum of f(x) = |3|x| + x + |4|x| 1| 4|.
- 3. (Answer: 516) How many triangles can you draw with vertices at integer coordinates on $[1,4]^2$?